

RESTRICTED LIE ALGEBRAS VIA MONADIC DECOMPOSITION

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ABSTRACT. We give a description of the category of restricted Lie algebras over a field \mathbb{k} of prime characteristic by means of monadic decomposition of the functor that computes the \mathbb{k} -vector space of primitive elements of a \mathbb{k} -bialgebra.

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INTRODUCTION

Let \mathbb{k} be a field and let \mathfrak{M} denote the category of \mathbb{k} -vector spaces. Denoting $\text{Alg}(\mathfrak{M})$ the category of unital, associative \mathbb{k} -algebras, there is the obvious forgetful functor $\Omega : \text{Alg}(\mathfrak{M}) \rightarrow \mathfrak{M}$, which has a left adjoint T . The composition ΩT defines a monad on \mathfrak{M} and the comparison functor Ω_1 from $\text{Alg}(\mathfrak{M})$ to ${}_{\Omega T}\mathfrak{M}$ -the Eilenberg-Moore category associated to the monad ΩT - can be shown to have a left adjoint T_1 such that the adjunction (T_1, Ω_1) becomes an equivalence of categories, i.e. Ω is a monadic functor.

It is well-known that for any $V \in \mathfrak{M}$, TV can be given moreover a \mathbb{k} -bialgebra structure, thus inducing a functor $\tilde{T} : \mathfrak{M} \rightarrow \text{Bialg}(\mathfrak{M})$. Now, a right adjoint for \tilde{T} is provided by the functor P that computes the space of primitive elements of any bialgebra. This adjunction furnishes \mathfrak{M} with a (different) monad $P\tilde{T}$. This time, P fails to be monadic, alas. Indeed, P_1 -the comparison functor associated to the monad $P\tilde{T}$ - still allows for a left adjoint \tilde{T}_1 , but the adjunction (\tilde{T}_1, P_1) is not an equivalence anymore. Yet, something can be done with it. Using the notation \mathfrak{M}_2 for the Eilenberg-Moore category of the monad $P_1\tilde{T}_1$ on ${}_{P\tilde{T}}\mathfrak{M}$ (the Eilenberg-Moore category of the monad $P\tilde{T}$), it was proven in [AGM] that there exists a functor

$$\tilde{T}_2 : \mathfrak{M}_2 \rightarrow \text{Bialg}(\mathfrak{M})$$

that allows a right adjoint P_2 and which is moreover full and faithful. This means that the functor P has so-called “monadic decomposition of length at most 2”.

In case the characteristic of the ground field \mathbb{k} is zero, the above result was further refined in [AMe2]. Indeed, amongst other things in the cited article, it is proven that the category \mathfrak{M}_2 is equivalent with $\text{Lie}(\mathfrak{M})$, the category of \mathbb{k} -Lie algebras. By the way, this is actually obtained as a consequence of a more general theorem (Theorem 7.2 in [AMe2]) that is proven for Lie algebras

2010 *Mathematics Subject Classification.* Primary 18C15; Secondary 16S30.

Key words and phrases. Monads, restricted Lie algebras.

This note was written while A. Ardizzoni was member of GNSAGA and partially supported by the research grant “Progetti di Eccellenza 2011/2012” from the “Fondazione Cassa di Risparmio di Padova e Rovigo”. He thanks the members of the department of Mathematics of both Vrije Universiteit Brussel and Université Libre de Bruxelles for their warm hospitality and support during his stay in Brussels in August 2013, when the work on this paper was initiated.

The second named author is a Marie Curie fellow of the Istituto Nazionale di Alta Matematica.

in abelian symmetric monoidal categories that satisfy certain extra conditions. In concreto, there exists a functor Γ such that $(P_2\tilde{U}, \Gamma)$ gives the afore-mentioned equivalence, \tilde{U} being the functor that computes the universal enveloping bialgebra of any Lie algebra in characteristic zero.

In case the characteristic of \mathbb{k} is a prime number p , things appear to be slightly different. Of course, one can still work with the ordinary definition of Lie algebra and consider its universal enveloping algebra (which is still a \mathbb{k} -bialgebra, also in finite characteristic), but this does not seem to be the appropriate notion if we wish to imitate the above-mentioned equivalence between \mathfrak{M}_2 and $\text{Lie}(\mathfrak{M})$ we had in case $\text{char}(\mathbb{k}) = 0$.

The aim of this note is to provide an appropriate equivalence in case of prime characteristic, using a slightly different approach than the one in [AMe2]. Therefore, recall that a *restricted Lie algebra* in characteristic p (which is a notion due to Jacobson, see [Jac]) is a triple $(L, [-, -], -^{[p]})$ where $(L, [,])$ is an ordinary Lie algebra, endowed with a map $-^{[p]} : L \rightarrow L$ satisfying three conditions. These restricted Lie algebras in many respects bear a closer relation to Lie algebras of characteristic 0 than ordinary Lie algebras of characteristic p . Now, restricted Lie algebras cannot be seen as Lie algebras in some abelian symmetric monoidal category, at least not to the authors' knowledge. However, in this short article we show that restricted Lie algebras allow for an interpretation using monadic decomposition of the functor P . Indeed, our main result states that one can construct a functor $\Lambda : \mathfrak{M}_2 \rightarrow \text{Lie}_p$, Lie_p being the category of restricted Lie algebras, such that $(P_2\tilde{u}, \Lambda)$ defines an equivalence between Lie_p and \mathfrak{M}_2 . Here \tilde{u} is the functor computing the restricted universal enveloping algebra of a restricted Lie algebra. (\tilde{T}_2, P_2) then turns out to identify with (\tilde{u}, \mathcal{P}) via Λ , \mathcal{P} being the functor that computes the restricted primitive elements of a bialgebra in characteristic p .

The article is organized as follows. In the preliminary section we recall some notation and results concerning monadic decomposition. Along the way, we address to the interested reader by stating two questions which seem to be of independent interest. In the second section, we consider adjoint squares, a powerful categorical tool that we can use in order to prove our main result, which is subject of the last section.

1. PRELIMINARY RESULTS

In this section, we shall fix some basic notation and terminology.

NOTATION 1.1. *Throughout this note \mathbb{k} will denote a field. \mathfrak{M} will denote the category of vector spaces over \mathbb{k} . Unadorned tensor product are to taken over \mathbb{k} unless stated otherwise.*

When X is an object in a category \mathcal{C} , we will denote the identity morphism on X by 1_X or X for short. For categories \mathcal{C} and \mathcal{D} , a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ will be the name for a covariant functor; it will only be a contravariant one if it is explicitly mentioned. By $\text{id}_{\mathcal{C}}$ we denote the identity functor on \mathcal{C} . For any functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we denote Id_F (or sometimes -in order to lighten notation in some computations- just F , if the context doesn't allow for confusion) the natural transformation defined by $\text{Id}_F X = 1_{FX}$.

1.2. Monadic decomposition. Recall that a *monad* on a category \mathcal{A} is a triple $\mathbb{Q} := (Q, m, u)$ consisting of a functor $Q : \mathcal{A} \rightarrow \mathcal{A}$ and natural transformations $m : QQ \rightarrow Q$ and $u : \mathcal{A} \rightarrow Q$ satisfying the associativity and the unitality conditions $m \circ mQ = m \circ Qm$ and $m \circ Qu = \text{Id}_Q = m \circ uQ$. An *algebra* over a monad \mathbb{Q} on \mathcal{A} (or simply a \mathbb{Q} -algebra) is a pair (X, μ) where $X \in \mathcal{A}$ and $\mu : QX \rightarrow X$ is a morphism in \mathcal{A} such that $\mu \circ Q\mu = \mu \circ mX$ and $\mu \circ uX = X$. A *morphism between two \mathbb{Q} -algebras* (X, μ) and (X', μ') is a morphism $f : X \rightarrow X'$ in \mathcal{A} such that $\mu' \circ Qf = f \circ \mu$. For the time being, we will denote by ${}_{\mathbb{Q}}\mathcal{A}$ the category of \mathbb{Q} -algebras and their morphisms. This is the so-called *Eilenberg-Moore category* of the monad \mathbb{Q} . We denote ${}_{\mathbb{Q}}U : {}_{\mathbb{Q}}\mathcal{A} \rightarrow \mathcal{A}$ the forgetful functor. When the multiplication and unit of the monad are clear from the context, we will just write Q instead of \mathbb{Q} .

Let $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$ be an adjunction with unit η and counit ϵ . Then $(RL, R\epsilon L, \eta)$ is a

monad on \mathcal{B} and we can consider the so-called *comparison functor* $K : \mathcal{A} \rightarrow {}_{RL}\mathcal{B}$ of the adjunction (L, R) which is defined by $KX := (RX, R\epsilon X)$ and $Kf := Rf$. Note that ${}_{RL}U \circ K = R$.

DEFINITION 1.3. An adjunction $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$ is called *monadic* (tripleable in Beck's terminology [Bec, Definition 3]) whenever the comparison functor $K : \mathcal{A} \rightarrow {}_{RL}\mathcal{B}$ is an equivalence of categories. A functor R is called *monadic* if it has a left adjoint L such that the adjunction (L, R) is monadic, see [Bec, Definition 3].

DEFINITION 1.4. ([AT, page 231]) A monad (Q, m, u) is called *idempotent* if m is an isomorphism. An adjunction (L, R) is called *idempotent* whenever the associated monad is idempotent.

The interested reader can find results on idempotent monads in [AT, MS]. Here we just note that the fact that (L, R) being idempotent is equivalent to requiring that ηR is a natural isomorphism. The notion of idempotent monad is tightly connected with the following.

DEFINITION 1.5. (See [AGM, Definition 2.7], [AHT, Definition 2.1] and [MS, Definitions 2.10 and 2.14]) Fix a $N \in \mathbb{N}$. A functor R is said to have a *monadic decomposition of monadic length N* when there exists a sequence $(R_n)_{n \leq N}$ of functors R_n such that

- 1) $R_0 = R$;
- 2) for $0 \leq n \leq N$, the functor R_n has a left adjoint functor L_n ;
- 3) for $0 \leq n \leq N - 1$, the functor R_{n+1} is the comparison functor induced by the adjunction (L_n, R_n) with respect to its associated monad;
- 4) L_N is full and faithful while L_n is not full and faithful for $0 \leq n \leq N - 1$.

A functor R having monadic length N is equivalent to requiring that the forgetful functor $U_{N, N+1}$ defines an isomorphism of categories and that no $U_{n, n+1}$ has this property for $n \leq N - 1$ (see [AGM, Remark 2.4]).

Note that for a functor $R : \mathcal{A} \rightarrow \mathcal{B}$ having a monadic decomposition of monadic length N , we thus have a diagram

$$(1) \quad \begin{array}{ccccccc} \mathcal{A} & \xleftarrow{\text{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\text{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\text{Id}_{\mathcal{A}}} \dots & \dots \xleftarrow{\text{Id}_{\mathcal{A}}} & \mathcal{A} \\ \uparrow L_0 & & \uparrow L_1 & & \uparrow L_2 & & & \uparrow L_N \\ \mathcal{B}_0 & \xleftarrow{U_{0,1}} & \mathcal{B}_1 & \xleftarrow{U_{1,2}} & \mathcal{B}_2 & \xleftarrow{U_{2,3}} \dots & \dots \xleftarrow{U_{N-1,N}} & \mathcal{B}_N \end{array}$$

where $\mathcal{B}_0 = \mathcal{B}$ and, for $1 \leq n \leq N$,

- \mathcal{B}_n is the category of $(R_{n-1}L_{n-1})$ -algebras ${}_{R_{n-1}L_{n-1}}\mathcal{B}_{n-1}$;
- $U_{n-1,n} : \mathcal{B}_n \rightarrow \mathcal{B}_{n-1}$ is the forgetful functor ${}_{R_{n-1}L_{n-1}}U$.

We will denote by $\eta_n : \text{id}_{\mathcal{B}_n} \rightarrow R_n L_n$ and $\epsilon_n : L_n R_n \rightarrow \text{id}_{\mathcal{A}}$ the unit, resp. counit of the adjunction (L_n, R_n) for $0 \leq n \leq N$. Note that one can introduce the forgetful functor $U_{m,n} : \mathcal{B}_n \rightarrow \mathcal{B}_m$ for all $m \leq n$ with $0 \leq m, n \leq N$.

We recall the following from [AGM]:

PROPOSITION 1.6. [AGM, Proposition 2.9] *Let $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$ be an idempotent adjunction. Then $R : \mathcal{A} \rightarrow \mathcal{B}$ has a monadic decomposition of monadic length at most 1.*

Letting \mathbb{k} be a field, and B a \mathbb{k} -bialgebra, the set $P(B)$ of primitive elements of B is defined as

$$P(B) = \{x \in B \mid \Delta(x) = 1 \otimes x + x \otimes 1\},$$

where Δ is the comultiplication of B . $P(B)$ forms a \mathbb{k} -vector space, yielding a functor

$$(2) \quad P : \text{Bialg}(\mathfrak{M}) \rightarrow \mathfrak{M}$$

Theorem 3.4 from loc. cit. asserts that the functor P has monadic decomposition at most 2, by showing that the comparison functor P_1 of the adjunction (\tilde{T}, P) admits a left adjoint \tilde{T}_1 such that the adjunction (\tilde{T}_1, P_1) is idempotent. For the sake of completeness, we recall here that \tilde{T} is the functor from \mathfrak{M} to $\text{Bialg}(\mathfrak{M})$, assigning to any vector space V the tensor algebra $T(V)$ (which can be endowed with a bialgebra structure $\tilde{T}(V)$, as is known).

Intriguingly, it is not known to the authors whether the bound provided by this above-mentioned Theorem 3.4 is sharp. It would thus be satisfying to have an answer to the following question -of independent interest- the interested reader is evidently invited to think about.

QUESTION 1.7. Is the functor \tilde{T}_1 fully faithful?

As mentioned in the Introduction, it is known -by combining Theorems 7.2 and 8.1 from [AMe2]- that in case $\text{char}(\mathbb{k}) = 0$, the category \mathfrak{M}_2 is equivalent to the category of \mathbb{k} -Lie algebras. It is the aim of this note to handle the case of finite characteristic. Before doing so, we would like to round off this preliminary section by the following.

DEFINITION 1.8. We say that a functor R is *comparable* whenever there exists a sequence $(R_n)_{n \in \mathbb{N}}$ of functors R_n such that $R_0 = R$ and, for $n \in \mathbb{N}$,

- 1) the functor R_n has a left adjoint functor L_n ;
- 2) the functor R_{n+1} is the comparison functor induced by the adjunction (L_n, R_n) with respect to its associated monad.

In this case we have a diagram as (1) but not necessarily stationary. Hence we can consider the forgetful functors $U_{m,n} : \mathcal{B}_n \rightarrow \mathcal{B}_m$ for all $m \leq n$ with $m, n \in \mathbb{N}$.

REMARK 1.9. Fix a $N \in \mathbb{N}$. A functor R having a monadic decomposition of monadic length N is comparable, see [AGM, Remark 2.10].

By the proof of Beck's Theorem [Bec, Proof of Theorem 1], one gets the following result.

LEMMA 1.10 ([AMe2]). *Let \mathcal{A} be a category such that, for any (reflexive) pair (f, g) ([BW, 3.6, page 98]) where $f, g : X \rightarrow Y$ are morphisms in \mathcal{A} , one can choose a specific coequalizer. Then the comparison functor $K : \mathcal{A} \rightarrow {}_{RL}\mathcal{B}$ of an adjunction (L, R) is a right adjoint. Thus any right adjoint $R : \mathcal{A} \rightarrow \mathcal{B}$ is comparable.*

Let \mathbb{k} again be a field of characteristic zero. Dually to \mathbb{k} -Lie algebras, recall that \mathbb{k} -Lie coalgebras, as introduced by Michaelis in [Mi1], are precisely Lie algebras in the abelian symmetric monoidal category $(\mathfrak{M}^{op}, \otimes^{op}, \mathbb{k})$, where associativity and unit constraints are taken to be trivial.

$Q(B)$ denotes the \mathbb{k} -Lie coalgebra of indecomposables of a \mathbb{k} -bialgebra, more precisely, $Q(B) = I/I^2$, where $I = \ker \varepsilon$, ε being the counit of B . This construction is functorial and, composed with the forgetful functor F from the category of \mathbb{k} -Lie coalgebras to \mathfrak{M} , yields the following functor

$$Q : \text{Bialg}(\mathfrak{M}) \rightarrow \mathfrak{M}.$$

In [Mi1, page 18], it is asserted that a right adjoint for the functor F is provided by the functor L^c that computes the so-called “cofree Lie coalgebra” on a vector space. Finally, Q has a right adjoint given by the cofree coalgebra functor \tilde{T}^c . In fact $Q = F \circ \tilde{Q}$, where \tilde{Q} is the functor sending a bialgebra B to the \mathbb{k} -Lie coalgebra $Q(B)$, while, by [Mi1, page 24], we have $\tilde{T}^c = \tilde{U}_H^c \circ L^c$, where $\tilde{U}_H^c(C)$ is the universal coenveloping bialgebra of a Lie coalgebra C (by the bialgebra version of [Mi1, Theorem, page 37], the functor \tilde{U}_H^c is right adjoint to \tilde{Q}). Lemma 1.10 now guarantees that the functor

$$Q^{op} : \text{Bialg}(\mathfrak{M}^{op}) \rightarrow \mathfrak{M}^{op}$$

is comparable. Strangely enough, at the moment, we don't have an answer to the following question, which is -again- of independent interest in the authors' opinion.

QUESTION 1.11. Does the functor Q^{op} also have monadic decomposition length at most 2? If so, is the resulting category \mathfrak{M}_2 equivalent to the category of \mathbb{k} -Lie coalgebras?

2. ADJOINT SQUARES

The main aim of this section is to extend some results on commutation data from the second section in the paper [AMe2]. This will be useful in the proof of our main result.

DEFINITION 2.1. Recall from [Gra, Definition I,6.7] that an *adjoint square* consists of a (not necessarily commutative) diagram of functors as depicted below $((L, R)$ and (L', R') being adjunctions with units η resp. η' and counits ϵ resp. ϵ') together with a matrix of natural transformations “inside”:

$$(3) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ L \updownarrow R & \left(\begin{array}{cc} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{array} \right) & L' \updownarrow R' \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array} \quad \begin{array}{ll} \zeta_{11} : L'G \rightarrow FL, & \zeta_{12} : L'GR \rightarrow F, \\ \zeta_{21} : G \rightarrow R'FL, & \zeta_{22} : GR \rightarrow R'F, \end{array}$$

These ingredients are required to be subject to the following equalities:

$$\begin{aligned} (4) \quad \zeta_{11} &= \zeta_{12}L \circ L'G\eta = \epsilon'FL \circ L'\zeta_{21} = \epsilon'FL \circ L'\zeta_{22}L \circ L'G\eta, \\ (5) \quad \zeta_{12} &= F\epsilon \circ \zeta_{11}R = \epsilon'F\epsilon \circ L'\zeta_{21}R = \epsilon'F \circ L'\zeta_{22}, \\ (6) \quad \zeta_{21} &= R'\zeta_{11} \circ \eta'G = R'\zeta_{12}L \circ \eta'G\eta = \zeta_{22}L \circ G\eta, \\ (7) \quad \zeta_{22} &= R'F\epsilon \circ R'\zeta_{11}R \circ \eta'GR = R'\zeta_{12} \circ \eta'GR = R'F\epsilon \circ \zeta_{21}R. \end{aligned}$$

We call such natural transformations *transposes* of each other. If only one of the entries of the matrix is given, its transposes can be defined by means of the equalities above.

EXAMPLE 2.2. Let $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$ be an adjunction with unit η and counit ϵ . Assume that the comparison functor $R_1 : \mathcal{A} \rightarrow \mathcal{B}_1$ has a left adjoint L_1 with unit η_1 and counit ϵ_1 . We then have an adjoint square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} \\ L_1 \updownarrow R_1 & \left(\begin{array}{cc} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{array} \right) & L \updownarrow R \\ \mathcal{B}_1 & \xrightarrow{G} & \mathcal{B} \end{array}$$

where

$$\pi_{11} \stackrel{(4)}{=} \epsilon L_1 \circ LU_{01}\eta_1 : LU_{01} \rightarrow L_1 \quad \text{and} \quad \pi_{22} = \text{id}_{U_{01}R_1}$$

so that, by the proof of [Bec, Theorem 1], π_{11} is the canonical projection defining L_1 . More explicitly, for every $(B, \mu) \in \mathcal{B}_1$ we have the following coequalizer of a reflexive pair in \mathcal{A}

$$LRLB \xrightleftharpoons[\epsilon_{LB}]{L\mu} LB = LU_{01}(B, \mu) \xrightarrow{\pi_{11}(B, \mu)} L_1(B, \mu).$$

As a consequence we get

$$\begin{aligned} \epsilon_1 \circ \pi_{11}R_1 &\stackrel{(5)}{=} \epsilon, \\ R\pi_{11} \circ \eta U_{01} &\stackrel{(6)}{=} U_{01}\eta_1. \end{aligned}$$

REMARK 2.3. Given two adjoint squares

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ L \updownarrow R & \left(\begin{array}{cc} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{array} \right) & L' \updownarrow R' \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{A}' & \xrightarrow{F'} & \mathcal{A}'' \\ L' \updownarrow R' & \left(\begin{array}{cc} \zeta'_{11} & \zeta'_{12} \\ \zeta'_{21} & \zeta'_{22} \end{array} \right) & L'' \updownarrow R'' \\ \mathcal{B}' & \xrightarrow{G'} & \mathcal{B}'' \end{array}$$

their horizontal composition is given by

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F'F} & \mathcal{A}'' \\ L \updownarrow R & \left(\begin{array}{cc} \zeta'_{11} * \zeta_{11} & \zeta'_{12} * \zeta_{12} \\ \zeta'_{21} * \zeta_{21} & \zeta'_{22} * \zeta_{22} \end{array} \right) & L'' \updownarrow R'' \\ \mathcal{B} & \xrightarrow{G'G} & \mathcal{B}'' \end{array}$$

where

$$\begin{aligned}\zeta'_{11} * \zeta_{11} &= F' \zeta_{11} \circ \zeta'_{11} G, \\ \zeta'_{12} * \zeta_{12} &= \zeta'_{12} \zeta_{12} \circ L'' G' \eta' G R, \\ \zeta'_{21} * \zeta_{21} &= R'' F' \epsilon' F L \circ \zeta'_{21} \zeta_{21}, \\ \zeta'_{22} * \zeta_{22} &= \zeta'_{22} F \circ G' \zeta_{22}.\end{aligned}$$

The following result should be compared with [Gra, Proposition 6.9].

LEMMA 2.4. *Consider an adjoint square as in the following diagram.*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ L \updownarrow R & \left(\begin{array}{cc} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{array} \right) & L' \updownarrow R' \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array}$$

Assume that F and G are equivalences of categories. Then the following assertions are equivalent.

- (1) ζ_{11} is an isomorphism.
- (2) ζ_{22} is an isomorphism.

Proof. Let $F' : \mathcal{A}' \rightarrow \mathcal{A}$ be a functor such that (F', F) is a category equivalence with (invertible) unit $\eta^{(F', F)}$ and counit $\epsilon^{(F', F)}$. Similarly we use the notation $\eta^{(G', G)}$ and $\epsilon^{(G', G)}$.

(2) \Rightarrow (1). Consider the following adjoint squares, where the diagrams on the right-hand side are obtained by rotating clockwise by 90 degrees the ones on the left-hand side (the upper index c stands for “clockwise”).

$$\begin{array}{ccc} \begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ L \updownarrow R & \left(\begin{array}{cc} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{array} \right) & L' \updownarrow R' \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array} & \xrightarrow{\circlearrowright} & \begin{array}{ccc} \mathcal{B} & \xrightarrow{L} & \mathcal{A} \\ G' \updownarrow G & \left(\begin{array}{cc} \zeta_{11}^c & \zeta_{12}^c \\ \zeta_{21}^c & \zeta_{22}^c = \zeta_{11} \end{array} \right) & F' \updownarrow F \\ \mathcal{B}' & \xrightarrow{L'} & \mathcal{A}' \end{array} \\ \\ \begin{array}{ccc} \mathcal{A} & \xrightarrow{R} & \mathcal{B} \\ F' \updownarrow F & \left(\begin{array}{cc} \zeta_{11}^r & \zeta_{12}^r \\ \zeta_{21}^r & \zeta_{22}^r = (\zeta_{11}^r)^{-1} \end{array} \right) & G' \updownarrow G \\ \mathcal{A}' & \xrightarrow{R'} & \mathcal{B}' \end{array} & \xrightarrow{\circlearrowright} & \begin{array}{ccc} \mathcal{A}' & \xrightarrow{F'} & \mathcal{A} \\ L' \updownarrow R' & \left(\begin{array}{cc} \zeta_{11}^{rc} & \zeta_{12}^{rc} \\ \zeta_{21}^{rc} & \zeta_{22}^{rc} = \zeta_{11}^r \end{array} \right) & L \updownarrow R \\ \mathcal{B}' & \xrightarrow{G'} & \mathcal{B} \end{array} \end{array}$$

Now apply [Gra, page 18], putting $(f, u) = (G', G)$, $(f', u') = (F', F)$, $(g, v) = (L, R)$, $(g', v') = (L', R')$, $\psi = \zeta_{22}$, $\tilde{\psi} = \zeta_{11}$, $\tilde{\tilde{\psi}} = \zeta_{11}^c$, $\theta = \zeta_{22}^r$, $\tilde{\theta} = \zeta_{11}^r$, $\tilde{\tilde{\theta}} = \zeta_{11}^{rc}$. Then we obtain that ζ_{11}^{rc} and ζ_{11}^c are mutual inverses. Note that

$$\zeta_{11} = \zeta_{22}^c \stackrel{(7)}{=} F L \epsilon^{(G', G)} \circ F \zeta_{11}^c G \circ \eta^{(F', F)} L' G$$

so that, as a composition of isomorphisms, ζ_{11} is an isomorphism.

(1) \Rightarrow (2). This implication is shown in a very similar fashion, by applying the dual result of [Gra, page 18]. □

DEFINITION 2.5. Following [BMW, 1.4], an adjoint square as in (3) is called *exact* whenever both $\zeta_{11} : L'G \rightarrow FL$ and $\zeta_{22} : GR \rightarrow R'F$ are isomorphisms. Note that this implies that the given diagram commutes -up to isomorphism- when either the left adjoint functors or the right adjoint functors are omitted.

REMARK 2.6. Consider a square of functors like in (3) and assume that $GR = R'F$. Then we can set $\zeta_{22} := \text{Id}_{GR}$ and we get an adjoint square. This square is exact if and only if $(F, G) : (L, R) \rightarrow (L', R')$ is a commutation datum in the sense of [AMe2, Definition 2.3].

3. THE CATEGORY OF RESTRICTED LIE ALGEBRAS

Fix an arbitrary field \mathbb{k} such that $\text{char}(\mathbb{k})$ is a prime p and recall that \mathfrak{M} denotes the category of vector spaces over \mathbb{k} .

DEFINITION 3.1. (due to Jacobson, see [Jac, page 210]) A *restricted Lie algebra over \mathbb{k}* (also called *p -Lie algebra* by some authors) is a triple $(L, [-, -], -^{[p]})$ consisting of a (ordinary) Lie algebra $(L, [-, -])$ (i.e. a \mathbb{k} -vector space L endowed with a \mathbb{k} -bilinear map $[-, -]$ satisfying the antisymmetry and Jacobi condition) and a (set-theoretical) map $-^{[p]} : L \rightarrow L$ satisfying

$$\begin{aligned} (\alpha x)^{[p]} &= \alpha^p x^{[p]} \text{ for all } x \in L, \alpha \in \mathbb{k}; \\ \text{ad}(x^{[p]}) &= (\text{ad}(x))^p \text{ for all } x \in L; \\ (x + y)^{[p]} &= x^{[p]} + y^{[p]} + s(x, y) \text{ for all } x, y \in L, \end{aligned}$$

where ad is the adjoint representation of L ;

$$\text{ad} : L \rightarrow \text{End}(L), x \mapsto \text{ad}_x \text{ where } \text{ad}_x(y) = [x, y],$$

and $s(x, y) = \sum_{i=1}^{p-1} \frac{s_i(x, y)}{i}$, where $s_i(x, y)$ is the coefficient of β^{i-1} in the expansion of $(\text{ad}(\beta x + y))^{p-1}(x)$.

A map $f : (L, [-, -], -^{[p]}) \rightarrow (L', [-, -]', -^{[p]'})$ is a morphism of restricted Lie algebras if f is a morphism of (ordinary) Lie algebras $f : (L, [-, -]) \rightarrow (L', [-, -]')$ such that $f(x^{[p]}) = (f(x))^{[p]'}$, for all $x \in L$.

The category of restricted Lie algebras with their morphisms will be denoted by Lie_p .

There is an adjunction $(\tilde{u} : \text{Lie}_p \rightarrow \text{Bialg}(\mathfrak{M}), \mathcal{P} : \text{Bialg}(\mathfrak{M}) \rightarrow \text{Lie}_p)$, given by the following functors (see [Gru] or [Mi2, Appendix], e.g.):

- $\tilde{u} : \text{Lie}_p \rightarrow \text{Bialg}(\mathfrak{M})$; the restricted universal enveloping algebra functor.
Explicitly, $\tilde{u}(L, [-, -], -^{[p]}) = \frac{\tilde{U}(L, [-, -])}{I}$, where I is the ideal in $\tilde{U}(L, [-, -])$ generated by elements of the form $x^p - x^{[p]}$.
- $\mathcal{P} : \text{Bialg}(\mathfrak{M}) \rightarrow \text{Lie}_p$; the restricted primitive functor.
Explicitly, for $B \in \text{Bialg}(\mathfrak{M})$, the space $P(B)$ becomes a Lie algebra for the commutator bracket $[-, -]$ and can moreover be endowed with the map $-^{[p]}$ sending an element $x \in L$ to x^p such that $\mathcal{P}(B) := (P(B), [-, -], -^{[p]})$ becomes a restricted Lie algebra.

We denote by $\tilde{\eta}_L$ the unit and by $\tilde{\epsilon}_L$ the counit of the adjunction (\tilde{u}, \mathcal{P}) . We also use the notation $H_{\text{Lie}_p} : \text{Lie}_p \rightarrow \mathfrak{M}$ for the forgetful functor. We obviously have that $H_{\text{Lie}_p} \mathcal{P} = P : \text{Bialg}(\mathfrak{M}) \rightarrow \mathfrak{M}$ is the usual primitive functor (cf. (2)).

Before stating the main result, we notice that in case we wish to stress the algebra nature of objects and morphisms in $\text{Alg}(\mathfrak{M})$, resp. the bialgebra nature of objects and morphisms in $\text{Bialg}(\mathfrak{M})$, we will do so by simply overlining, resp. over and underlining things. Please mind as well that we denote η (resp. $\tilde{\eta}$) and ϵ (resp. $\tilde{\epsilon}$) the unit and counit of the adjunction (T, Ω) (resp. (\tilde{T}, P)).

THEOREM 3.2. *We have the following diagram.*

$$(8) \quad \begin{array}{ccccc} \text{Bialg}(\mathfrak{M}) & \xleftarrow{\text{id}_{\text{Bialg}(\mathfrak{M})}} & \text{Bialg}(\mathfrak{M}) & \xleftarrow{\text{id}_{\text{Bialg}(\mathfrak{M})}} & \text{Bialg}(\mathfrak{M}) \\ \tilde{T} \downarrow P & \swarrow \text{id}_{\text{Bialg}(\mathfrak{M})} & \tilde{T}_1 \downarrow P_1 & \swarrow \text{id}_{\text{Bialg}(\mathfrak{M})} & \tilde{T}_2 \downarrow P_2 \\ \mathfrak{M} & \xleftarrow{U_{0,1}} & \mathfrak{M}_1 & \xleftarrow{U_{1,2}} & \mathfrak{M}_2 \\ & \searrow H_{\text{Lie}_p} & & \searrow \Lambda & \\ & & \text{Lie}_p & & \end{array}$$

The functor P is comparable so that we can use the notation of Definition 1.8. There is a functor $\Lambda : \mathfrak{M}_2 \rightarrow \text{Lie}_p$ such that $\Lambda \circ P_2 = \mathcal{P}$ and $H_{\text{Lie}_p} \circ \Lambda = U_{0,2}$.

- The adjunction (\tilde{T}_1, P_1) is idempotent, we can choose $\tilde{T}_2 := \tilde{T}_1 U_{1,2}$, $\pi_2 = \text{Id}_{\tilde{T}_2}$ and \tilde{T}_2 is full and faithful i.e. $\tilde{\eta}_2$ is an isomorphism. The functor P has a monadic decomposition of monadic length at most two.
- $(\text{id}_{\text{Bialg}(\mathfrak{M})}, \Lambda) : (\tilde{T}_2, P_2) \rightarrow (\tilde{\mathfrak{u}}, \mathcal{P})$ is a commutation datum (with corresponding matrix (χ_{ij})).
- The pair $(P_2 \tilde{\mathfrak{u}}, \Lambda)$ is an equivalence of categories with unit $\tilde{\eta}_L$ and counit $(\tilde{\eta}_2)^{-1} \circ P_2 \chi_{11}$. Thus (\tilde{T}_2, P_2) identifies with $(\tilde{\mathfrak{u}}, \mathcal{P})$ via Λ .

Proof. By [AGM, Theorem 3.4], the functor P has monadic decomposition of monadic length at most 2. Moreover, the adjunction (\tilde{T}_1, P_1) is idempotent and we can define a functor $\Lambda : \mathfrak{M}_2 \rightarrow \text{Lie}_{\mathfrak{p}}$. Indeed, letting $V_2 = ((V_0, \mu_0), \mu_1)$ be an object in \mathfrak{M}_2 , we can define an object $\Lambda V_2 \in \text{Lie}_{\mathfrak{p}}$ as follows:

$$\Lambda V_2 = \left(V_0, [-, -], -^{[p]} \right),$$

where $[-, -] : V_0 \otimes V_0 \rightarrow V_0$ is defined by setting $[x, y] := \mu_0(x \otimes y - y \otimes x)$, for every $x, y \in V_0$, while $-^{[p]} : V_0 \rightarrow V_0$ is defined by setting $x^{[p]} := \mu_0(x^{\otimes p})$, for every $x \in V_0$.

Let $f_2 : V_2 \rightarrow V_2'$ be a morphism in \mathfrak{M}_2 and set $f_1 := U_{1,2} f_2$ and $f_0 := U_{0,1} f_1$. Then, for every $x, y \in V_0$

$$\begin{aligned} f_0([x, y]) &= f_0 \mu_0(x \otimes y - y \otimes x) = \mu'_0(P \tilde{T} f_0)(x \otimes y - y \otimes x) \\ &= \mu'_0\left(\left(\tilde{T} f_0\right)(x \otimes y - y \otimes x)\right) = \mu'_0[(f_0(x) \otimes f_0(y) - f_0(y) \otimes f_0(x))] \\ &= [f_0(x), f_0(y)]' \end{aligned}$$

and

$$\begin{aligned} f_0(x^{[p]}) &= f_0 \mu_0(x^{\otimes p}) = \mu'_0(P \tilde{T} f_0)(x^{\otimes p}) = \mu'_0\left(\left(\tilde{T} f_0\right)(x^{\otimes p})\right) \\ &= \mu'_0\left(f_0(x)^{\otimes p}\right) = f_0(x)^{[p]}. \end{aligned}$$

Thus f_2 induces a unique morphism Λf_2 such that $H_{\text{Lie}_{\mathfrak{p}}}(\Lambda f_2) = f_0$. This defines a functor $\Lambda : \mathfrak{M}_2 \rightarrow \text{Lie}_{\mathfrak{p}}$, as claimed above. By construction $H_{\text{Lie}_{\mathfrak{p}}} \circ \Lambda = U_{0,2}$. Moreover,

$$(\Lambda \circ P_2)(\overline{B}) = \Lambda(P_2(\overline{B})).$$

In order to proceed, we have to compute $\Lambda(P_2(\overline{B}))$. We have that

$$P_1(\overline{B}) = \left(P(\overline{B}), P \tilde{\epsilon}_{\overline{B}} : P \tilde{T} P(\overline{B}) \rightarrow P(\overline{B}) \right)$$

so the brackets $[-, -]$ and $-^{[p]}$ are, for every $x, y \in P(\overline{B})$, given by the following:

$$\begin{aligned} [x, y] &= \left(P \tilde{\epsilon}_{\overline{B}} \right)(x \otimes y - y \otimes x) = \left(\tilde{\epsilon}_{\overline{B}} \right)(x \otimes y - y \otimes x) = (\epsilon_{\overline{B}})(x \otimes y - y \otimes x) = xy - yx, \\ x^{[p]} &= \left(P \tilde{\epsilon}_{\overline{B}} \right)(x^{\otimes p}) = \left(\tilde{\epsilon}_{\overline{B}} \right)(x^{\otimes p}) = (\epsilon_{\overline{B}})(x^{\otimes p}) = x^p \end{aligned}$$

so that $(\Lambda \circ P_2)(\overline{B}) = \mathcal{P}(\overline{B})$. For the morphisms, we have

$$(H_{\text{Lie}_{\mathfrak{p}}} \Lambda P_2)(\overline{f}) = U_{0,2} P_2(\overline{f}) = P(\overline{f}) = (H_{\text{Lie}_{\mathfrak{p}}} \mathcal{P})(\overline{f}).$$

Since $H_{\text{Lie}_{\mathfrak{p}}}$ is faithful, we conclude that $\Lambda P_2(\overline{f}) = \mathcal{P}(\overline{f})$ and hence $\Lambda \circ P_2 = \mathcal{P}$.

Now, since the adjunction (\tilde{T}_1, P_1) is idempotent, by [AGM, Proposition 2.3], we can choose $\tilde{T}_2 := \tilde{T}_1 U_{1,2}$ with $\tilde{\eta}_1 U_{1,2} = U_{1,2} \tilde{\eta}_2$ and $\tilde{\epsilon}_1 = \tilde{\epsilon}_2$.

We have an adjoint square

$$\begin{array}{ccc} \text{Bialg}(\mathfrak{M}) & \xrightarrow{\text{id}_{\text{Bialg}(\mathfrak{M})}} & \text{Bialg}(\mathfrak{M}) \\ \tilde{T}_2 \updownarrow P_2 & \left(\begin{array}{cc} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{array} \right) & \tilde{\mathfrak{u}} \updownarrow \mathcal{P} \\ \mathfrak{M}_2 & \xrightarrow{\Lambda} & \text{Lie}_{\mathfrak{p}} \end{array}$$

where $\chi_{22} = \text{Id}_{\Lambda P_2} : \Lambda P_2 \rightarrow \mathcal{P}$. We have

$$(9) \quad \mathcal{P}\chi_{11} \circ \tilde{\eta}_L \Lambda \stackrel{(6)}{=} \Lambda \tilde{\eta}_2.$$

Since \tilde{T}_2 is full and faithful, we have that $\tilde{\eta}_2$ is an isomorphism. By [MM, Theorem 6.11(1)], we also know that $\tilde{\eta}_L : \text{Id}_{\text{Lie}_p} \rightarrow \mathcal{P}\tilde{\mathbf{u}}$ is an isomorphism. Thus $\mathcal{P}\chi_{11}$ is an isomorphism.

Let us check that $(P_2\tilde{\mathbf{u}}, \Lambda)$ is an adjunction with unit $\tilde{\eta}_L$ and counit $(\tilde{\eta}_2)^{-1} \circ P_2\chi_{11}$. We have

$$\begin{aligned} \Lambda \left((\tilde{\eta}_2)^{-1} \circ P_2\chi_{11} \right) \circ \tilde{\eta}_L \Lambda &= \Lambda (\tilde{\eta}_2)^{-1} \circ \Lambda P_2\chi_{11} \circ \tilde{\eta}_L \Lambda \\ &= (\Lambda \tilde{\eta}_2)^{-1} \circ \mathcal{P}\chi_{11} \circ \tilde{\eta}_L \Lambda \stackrel{(9)}{=} (\Lambda \tilde{\eta}_2)^{-1} \circ \Lambda \tilde{\eta}_2 = \text{Id}_\Lambda \end{aligned}$$

and

$$\begin{aligned} \left((\tilde{\eta}_2)^{-1} \circ P_2\chi_{11} \right) P_2\tilde{\mathbf{u}} \circ P_2\tilde{\mathbf{u}}\tilde{\eta}_L &= (\tilde{\eta}_2 P_2\tilde{\mathbf{u}})^{-1} \circ P_2\chi_{11} P_2\tilde{\mathbf{u}} \circ P_2\tilde{\mathbf{u}}\tilde{\eta}_L \\ &= P_2\tilde{\epsilon}_2\tilde{\mathbf{u}} \circ P_2\chi_{11} P_2\tilde{\mathbf{u}} \circ P_2\tilde{\mathbf{u}}\tilde{\eta}_L \\ &= P_2((\tilde{\epsilon}_2 \circ \chi_{11} P_2)\tilde{\mathbf{u}} \circ \tilde{\mathbf{u}}\tilde{\eta}_L) \\ &\stackrel{(5)}{=} P_2((\tilde{\epsilon}_L \circ \tilde{\mathbf{u}}\chi_{22})\tilde{\mathbf{u}} \circ \tilde{\mathbf{u}}\tilde{\eta}_L) \\ &= P_2(\tilde{\epsilon}_L\tilde{\mathbf{u}} \circ \tilde{\mathbf{u}}\tilde{\eta}_L) = \text{Id}_{P_2\tilde{\mathbf{u}}}. \end{aligned}$$

Using that $\Lambda \circ P_2 = \mathcal{P}$ and $H_{\text{Lie}_p} \circ \Lambda = U_{0,2}$, we get

$$U_{0,2} P_2\chi_{11} = (H_{\text{Lie}_p}) \Lambda P_2\chi_{11} = (H_{\text{Lie}_p}) \mathcal{P}\chi_{11}.$$

Since the latter is an isomorphism and since $U_{0,2}$ is conservative, we deduce that $P_2\chi_{11}$ is an isomorphism too. We have thus proved that $(P_2\tilde{\mathbf{u}}, \Lambda)$ is a category equivalence. By Lemma 2.4, from $\chi_{22} = \text{Id}_{\Lambda P_2}$, we deduce that χ_{11} is an isomorphism, too. \square

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